2006

MS 221 MOCK EXAM

PART I

Instructions

(i) *You should attempt as many questions as you can in this part of the examination.*

(ii) *Part 1 carries 72% of the available examination marks. Each question carries an indication of the number of marks that are allocated to it.*

(iii) *You should record your answers to each question in the answer book(s) provided. You are strongly advised to show all your working, including any rough working.*

Question 1 - 6 marks

(a) Use the Fibonacci recurrence relation to show that

$$\frac{F_n}{F_{n+1}F_{n+2}} = \frac{1}{F_{n+1}} - \frac{1}{F_{n+2}} \qquad \text{for } n = 0, 1, 2\ldots$$ [3]

(b) Hence use the method of telescoping cancellation to show that

$$\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{2}{3\times5} + \frac{3}{5\times8} + \ldots \frac{F_n}{F_{n+1}F_{n+2}} = 1 - \frac{1}{F_{n+2}} \qquad \text{for } n = 0, 1, 2\ldots$$ [3]

Question 2 - 6 marks

This question concerns the curve with equation

$$9x^2 + 4y^2 - 36x + 8y + 4 = 0 \quad \text{(equation 1)}$$

(a) Show that this curve can be obtained from the conic with equation

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

by translation, and state the translation required. [3]

(b) Hence sketch the curve with equation 1, showing its axes of symmetry and the coordinates of its vertices. [3]

Question 3 - 6 marks

The isometry g is defined to be the reflection q_θ in the line $y=\frac{3x}{4}$ followed by the translation $t_{8,6}$.

(a) Show that g is a glide-reflection. [1]

(b) Give the rules for q_θ and $t_{8,6}$; hence determine the rule for g. [3]

(c) Determine the image of each of the points (0, 0), (0, 1), (1, 0) and (1, 1) under g. [2]

Question 4 - 5 marks

From a group of 9 people, of whom 5 are men and 4 are women, 5 are to be selected to form a committee.

(a) In how many different ways may this selection be made? [2]

(b) In what proportion of this total number of possible selections will women outnumber men on the committee? [3]

Question 5 - 7 marks

(a) Find the fixed points of the function $f(x) = 2x(1 - x)$ where $x \in \mathbb{R}$. [2]

(b) Draw a sketch of the graph of f, together with the line $y = x$, showing the fixed points, and hence explain whether each fixed point is attracting or repelling. [2]

(c) Use the gradient criterion to find an interval of attraction for one of the fixed points. [3]

Question 6 - 6 marks

Express the matrix M below in the form QDQ^{-1}, where D is a diagonal matrix (you should calculate the matrix Q^{-1} explicitly):

$$M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$ [6]

Question 7 – 4 marks

Differentiate the following functions. In each case, state which of the principal rules of calculus you are using.

(a) $k(x) = \ln(\arcsin(x)) \quad (0 < x < 1)$ [2]

(b) $g(t) = t^3 \cos(3t)$ [2]

Question 8 - 5 marks

(a) Using integration by parts, find the indefinite integral

$$\int x^3 \ln(3x)\,dx \quad (x > 0).$$ [2]

(b) Using the substitution $u = \ln(x)$, or otherwise, find the indefinite integral

$$\int \frac{\sec^2(\ln(x))}{x}\,dx.$$ [3]

Question 9 - 4 marks

(a) Find the volume of revolution obtained when the region under the graph of

$y = \dfrac{1}{x}$, from x = 1 to x = k (where k > 1 is a constant), is rotated about the x-axis. [3]

(b) Hence show that this volume tends to a finite limit as $k \to \infty$. [1]

Question 10 - 5 marks

(a) Use appropriate results from the Handbook to find the first three non-zero terms of the Taylor series about $x = 0$ for the function

$$f(x) = \sin(x)\cos(x).$$ [2]

(b) By differentiating term by term your answer to (a), show that the Taylor polynomial of degree 4 for $f'(x)$ is the same as that for cos(2x), and explain why you would expect this to be the case. [3]

Question 11 - 5 marks

Let $z = 8\exp\left(\dfrac{i\pi}{3}\right)$.

(a) Express z and $\bar{z}$ in Cartesian form, where $\bar{z}$ is the complex conjugate of z. [3]

(b) Hence, find in Cartesian form $z + \bar{z}$ and $z\bar{z}$. [2]

Question 12 - 4 marks

(a) Show that 658 324 719 is divisible by 9 but not divisible by 18. [2]

(b) Find a number x in $\mathbb{Z}_{15}$ such that $x \times_{15} 4 = 5$. [1]

(c) Give an example of a number x in $\mathbb{Z}_{15}$ which has no multiplicative inverse. [1]

Question 13 - 4 marks

Consider the group $(G, *)$ whose Cayley table is given below.

$*$	p	q	r	s
p	r	s	p	q
q	s	r	q	p
r	p	q	r	s
s	q	p	s	r

(a) Which element is the identity element of $(G, *)$? [1]

(b) Write down all the self-inverse elements of $(G, *)$. [1]

(c) Is $(G, *)$ Abelian ? Briefly explain your answer. [1]

(d) To which of the groups listed in the Handbook is $(G, *)$ isomorphic ? [1]

Question 14 - 5 marks

Here are two statements about integers n and m, only one of which is true.

(A) If n and m are both even then $n^2 - m^2$ is even.

(B) If $n^2 - m^2$ is even then n and m are both even.

(a) Which of these statements is false ? [1]

(b) Prove that the statement you have identified in (a) is false. [3]

(c) What is the name of the style of proof you have used in (b) ? [1]

PART II

Instructions

(i) *You should attempt not more than* **TWO** *questions from this part of the examination.*

(ii) *Each question in this part carries 14% of the marks.*

(iii) *You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.*

(iv) *Show all your workings.*

Question 15

(a) Find a closed form for the sequence given by the following recurrence system:

$u_0 = 1,\ u_1 = 2,\ \ u_{n+2} = -4u_{n+1} - 4u_n \qquad (n = 0, 1, 2, \ldots)$ [6]

(b) Show that this sequence satisfies the identity

$u_{n+1}u_{n-1} - u_n{}^2 = -4^{n+1} \qquad (n = 1, 2, 3, \ldots)$ [4]

(c) Show that the sequence also satisfies the identity

$$\frac{u_{n+1}}{u_n} = \frac{2+4n}{1-2n} \qquad (n = 0,\ 1,\ 2\ldots)$$

and deduce that $\dfrac{u_{n+1}}{u_n} \to -2$ as $n \to \infty$. [4]

Question 16

(a) Find, in the form $\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x} + \boldsymbol{a}$ where $\boldsymbol{A}$ is a 2x2 matrix and $\boldsymbol{a}$ is a vector with two components, the rule of the affine transformation $f : \square^2 \to \square^2$ that maps the points (0, 0), (1, 0) and (0, 1) to the points (3, 2), (5, 1) and (6, 0) respectively. [3]

(b) Hence find the area of the triangle with vertices (3, 2), (5, 1) and (6, 0). [1]

(c) Show that the matrix $\boldsymbol{A}$ that you found in part (a) is self-inverse (that is, that $\boldsymbol{A} = \boldsymbol{A}^{-1}$). [2]

(d) Let $\boldsymbol{x}_1 = \boldsymbol{A}\boldsymbol{x}_0 + \boldsymbol{a}$, where $\boldsymbol{A}$ and $\boldsymbol{a}$ are the matrix and vector from part (a) and $\boldsymbol{x}_0$ is an arbitrary two-component vector. Use the result of part (b) to show that $\boldsymbol{x}_0 = \boldsymbol{A}(\boldsymbol{x}_1 - \boldsymbol{a})$. [3]

(e) Use the result of part (d) to find, in the form $\boldsymbol{x} \mapsto \boldsymbol{B}\boldsymbol{x} + \boldsymbol{b}$ where $\boldsymbol{B}$ is a 2x2 matrix and $\boldsymbol{b}$ is a two-component vector, the rule of the affine transformation f^{-1}. [2]

(f) Hence find the images under f of the lines $y = x$ and $y = -x$. [3]

Question 17

Use the graph-sketching strategy of Chapter C1 to sketch the graph of the real function $f(x) = \dfrac{2x+1}{2(3x+1)(x-1)}$. [14]

Question 18

(a) Prove, using mathematical induction, that

$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$$

for all positive integers n.

[You may assume, without proof, the formula $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$.] [8]

(b) Use Fermat's Little Theorem to find the remainder when 221^{2004} is divided by 19.

[6]

[END OF QUESTION PAPER]